FROBENIUS MAPS AND ALGEBRAIC VARIETIES

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ABSTRACT. Many characterization of abelian varieties in positive characteristic is given by several authors. In this article, we give a characterization of abelian varieties by the language of $F_*\mathcal{O}_X$ and its Poincare bundles.

Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth proper variety over k. When does X satisfy the following property $(*)$?

$(*)$ $F_*\mathcal{O}_X \simeq \bigoplus_j M_j$ where $F : X \to X$ is the absolute Frobenius morphism and each M_j is a line bundle.

For example, an arbitrary smooth proper toric variety satisfies this property $(*)$ (cf. [Achinger][Thomsen]). Thus there are many varieties which satisfy $(*)$. But every toric variety has negative Kodaira dimension. On the other hand,In [ST] we show that ordinary abelian varieties satisfy $(*)$ and this property gives the characterization of the ordinary abelian varieties.

Theorem 0.1. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth projective variety over k. Assume the following conditions.*

- *For infinitely many $e \in \mathbb{Z}_{>0}$, $F^e_*\mathcal{O}_X \simeq \bigoplus_j M_j$ where each M_j is an invertible sheaf.*
- *$\kappa(X) \geq 0$ where $\kappa(X)$ is the Kodaira dimension of X.*

Then X is an ordinary abelian variety.

By the Theorem above, it is natural to ask the same criterion holds only for F_*O_X. But in [ES], we gave the counter example for that problem.

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Theorem 0.2. *Let X be a non-abelian smooth projective surface. Then X satisfies the following conditions if and only if X is an F-split Igusa surface:*

- *$F_*\mathcal{O}_X$ is isomorphic to direct sum of line bundles.*
- *K_X is pseudo-effective.*

We also showed that if we consider $e > 1$, then the same characterization holds.

Theorem 0.3. *Let X be a smooth projective variety over k of characteristic $p > 2$ (resp. $p > 0$). X is an ordinary abelian variety if and only if the following conditions hold:*

- *$F_*\mathcal{O}_X$ (resp. $F^2_*\mathcal{O}_X$) is isomorphic to a direct sum of line bundles.*
- *K_X is pseudo-effective.*

Another natural question is how can we characteraze non ordinary abelian varieties by the language of $F_*\mathcal{O}_X$. In this article we give the characterization by using Poincare bundle on albanese varieties. For details of Poincare bundles and albanese maps, see [FGAex, Section 9].

Theorem 0.4. *Let X be a smooth projective variety over k of characteristic p. There is a canonical map $\Phi : \tilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$ with the following properties:*

- *$\widetilde{\mathcal{P}}_X \simeq p_{1*}\mathcal{P}_{X\times kerV}$ where $\mathcal{P}$ is the Poincare bundle on $X \times Pic^0X$ and V is the Verschiebung map on Pic^0X.*
- *Φ is isomorphic if and only if X is an abelian variety.*
- *Φ is generically isomorphic if and only if X has maximal albanese dimension.*

We overview the proof of Theorem 0.4. At first, we construct the map $\Phi : \tilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$.

Lemma 0.5. *Let X be a smooth projective variety over k of characteristic p and $\alpha_X : X \to \mathrm{Alb}(X)$ be the albanese map of X. Then we have*

$$\alpha^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \widetilde{\mathcal{P}}_X$$

Proof. $\alpha_X : X \to \mathrm{Alb}(X)$ induces the following map;

$$\alpha_X^* : \mathrm{Pic}^0\mathrm{Pic}^0\mathrm{Pic}^0_{red}(X) \simeq \mathrm{Pic}^0(X)$$

Here we can see that the kernel of the Verschiebung maps corresponds each other. Hence, we have

$$\alpha_X^*\mathrm{KerV}_{\mathrm{Pic}^0(X)} \simeq \mathrm{KerV}_{\mathrm{Pic}^0\mathrm{Pic}^0\mathrm{Pic}^0_{red}(X)}$$

. By considering the universal bundles on them, we get the assersion. □

Lemma 0.6. *Let X be a smooth projective variety over k of characteristic p. There is a canonical map $\Phi : \widetilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$.*

Proof. We use the following theorem.

Theorem 0.7 (Oda)**.** *Let $f : X \to Y$ be an isogeny of abelian varieties over k. Set $\hat{f} : \hat{Y} \to \hat{X}$ to be the dual of f. Let $L \in \mathrm{Pic}^0(X)$. Then, f*

$$f_*L \simeq \mathrm{pr}_{1*}(\mathcal{P}_Y|_{Y \times \hat{f}^{-1}([L])})$$

where $\mathcal{P}_Y$ is the normalized Poincare line bundle of $(Y, 0)$*.*

By applying this theorem to $f = F$ and $L = \mathcal{O}_X$, we have $\Phi : \widetilde{\mathcal{P}}_X \to F_*\mathcal{O}_X$ for any abelian varieties X. (Note that Verschiebung map is defined by the dual of relative Frobenius map.)
In general, we have the base change map $\alpha^* F_*\mathcal{O}_{\mathrm{Alb}(X)} \to F_*\mathcal{O}_X$. Then by the discussion above and lem 0.6,

$$\alpha^* F_*\mathcal{O}_{\mathrm{Alb}(X)} \simeq \alpha^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \widetilde{\mathcal{P}}_X$$

Hence, by composing them, we get the assersion. □

Claim 0.8. *If Φ_X is isomorphism, then X has maximal albanese dimension, namely* $\dim X = \dim \alpha(X)$

Proof. By Stein factrizaton theorem, we have the following decomposition;

$$\alpha : X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} A = \mathrm{Alb}(X)$$

where g, h is finite, and g is purely inseparable, h is separable, Y, Z is normal.

We first see the map

$$\Phi_Y : \widetilde{\mathcal{P}_Y} \to F_*\mathcal{O}_Y$$

. Pulling back by f, we have

$$\widetilde{\Phi}_Y : f^*\widetilde{\mathcal{P}}_Y \overset{f^*\Phi_Y}{\to} f^*F_*\mathcal{O}_Y \to F_*\mathcal{O}_X$$

where the second map is base change map.
We also have

$$f^*\widetilde{\mathcal{P}}_Y \simeq f^*g^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \alpha^*\widetilde{\mathcal{P}}_{\mathrm{Alb}(X)} \simeq \widetilde{\mathcal{P}}_X.$$

Hence, we see that $\widetilde{\Phi}_Y = \Phi_X$. Hence, by calculating the rank of $F_*\mathcal{O}$, we have $\dim X \le \dim Y$. This implies $\dim X = \dim Y = \dim \alpha(X)$. □

Claim 0.9. *If Φ_X is isomorphism, then Φ_Y is also isomorphism.*

Proof. Since X has maximal albanese dimension, f is a birational contraction. Let U, V be the isomophic locus of f in X, Y respectively. Then U, V have codimension two in their ambient space. Here we have

$$F_*\mathcal{O}_Y|U \simeq F_*\mathcal{O}_X|V \simeq \mathcal{P}_X|V \simeq \mathcal{P}_Y|U$$

Since all the sheaf above are reflexive and all the varieties above are normal, and U, V are codimension two subset, we get $\Phi_Y : \widetilde{\mathcal{P}_Y} \simeq F_*\mathcal{O}_Y$. □

Claim 0.10. *There is a decomposition*

$$Y \to Z_1 \to ...Z_i \to Z_{i+1}... \to Z$$

such that $h_i : Z_i \to Z_{i+1}$ is a purely inseparable extension of height 1 of normal varieties. Futhermore, if Φ_Y is an isomorphism, then Φ_{Z_i} is also an isomorphism.

Proof. The first assersion follows by taking field extensions of height 1 and normalizations in the fields. We see the second assersion. Since Z_i is normal, $F_*\mathcal{O}_{Z_i}$ is reflexive. Hence we may assume Z_{i+1} is a spectrum of a DVR. Since torsion free module over DVR is faithfully flat, h_i^* is a faithfully flat functor. Hence by the induction starting from Y, we have the second assersion. □

By standard calculation of Grothendiek duality and Kunz's theorem, we have the following proposition.

Proposition 0.11. *Let X be normal projective variety over k. Assume that Φ_X is an isomorphism. Then X is smooth and $-(p-1)K_X$ is an effective divisor.*

We prove the main theorem for purelu inseparable extension of height 1.

Proposition 0.12. *Let $\alpha : Z \to A$ be the albanese map of normal variety Z. Assume that α is purely inseparable of height 1 and Φ_Z is an isomorphism. Then α is an isomorphism.*

Proof. By [Eke], any purely inseparable of height 1 morphism can be described by a quotient of 1-foliation. Namely, there is a 1-foliation $\mathcal{F} \subset \mathcal{T}_A$ such that $Z \simeq A/\mathcal{F}$ and $\omega_Z^p \simeq \alpha^*\omega_A \otimes (det\mathcal{F})^{(1-p)}$ Since A is an abelian variety, $\mathcal{T}_A$ is trivial. This implies $det\mathcal{F} \subset \oplus\mathcal{O}_A$. Hence $det\mathcal{F})$ is negative line bundle, put $det\mathcal{F} \simeq \mathcal{O}(-D)$ for an effective divisor D. Then we have

$$\omega_Z^p \simeq \mathcal{O}_Z((p-1)D).$$

Prop 0.11 and this implies $\kappa(Z) = 0$. Hence by [HP], we get the assersion.

$\square$

By claim 0.10 and Prop 0.12, h is an isomorphism. Next, we show g is an isomorphism.

Claim 0.13. *g is an isomorphism.*

Proof. Put

$$K_Y = g^* K_A + R$$

where R is the ramification divisor of separable morphism g. Since $K_A = 0$ and by Prop 0.11, we have $R = 0$.

Hence $\alpha : Y \to A$ is etale in codimension one. Then, by the Zariski–Nagata purity, α is etale. By [Mumford, Section 18, Theorem], Y is also an abelian variety and we are done.

$\square$

Claim 0.14. *f is an isomorphism*

Proof. We can write

$$K_X = f^* K_A + E$$

where E is an f-exceptional divisor. Since A is terminal (cf. [KM, Section 2.3]), E is effective. By Prop 0.12, we have $K_X \equiv 0$. Since $K_X \equiv 0$, we see that E is f-nef. By the negativity lemma (cf. [KM, Lemma 3.39]), we see $E = 0$. Thus, the codimension of $\mathrm{Ex}(f)$ in X is at least two. Since A is smooth, f is an isomorphism. $\square$

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